

# Symmetries of $N = 4$ supersymmetric $\mathbb{CP}^n$ mechanics

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## Abstract

We explicitly constructed the generators of  $SU(n+1)$  group which commute with the supercharges of  $N = 4$  supersymmetric  $\mathbb{CP}^n$  mechanics in the background  $U(n)$  gauge fields. The corresponding classical Hamiltonian can be represented as a direct sum of two Casimir operators: one Casimir operator on  $SU(n+1)$  group contains our bosonic and fermionic coordinates and momenta, while the second one, on the  $SU(1, n)$  group, is constructed from isospin degrees of freedom only.

# 1 Introduction

The construction of supersymmetric non-linear sigma-models in one dimensions has been known for many years [1, 2, 3]. If the scalar fields take values in a complex Kähler manifold, then the supersymmetric Lagrangian is given by an elegant and simple expression in terms of the Kähler metric. In the simplest case of the supersymmetric  $\mathbb{CP}^n$  model [4] in one dimension the  $N = 4$  supercharges acquire, in the appropriate basis, a very simple form [5, 6, 7]. The main properties of the model can be described by the statement that the supercharges and the Hamiltonian are invariant under the action of the  $SU(n+1)$  group, whereas the bosonic component fields parameterize the  $SU(n+1)/U(n)$  manifold [8].

One of the interesting further questions is how to introduce the interaction in the supersymmetric  $\mathbb{CP}^n$  model without breaking the  $SU(n+1)$  symmetry. Clearly, the evident guess is to try to introduce the interaction with non-Abelian background fields living on a  $U(n)$  subgroup. In the bosonic case such a system has been proposed by Karabali and Nair who constructed higher dimensional quantum Hall systems on  $\mathbb{CP}^n$  manifolds [9].<sup>1</sup> The corresponding bulk and edge actions were derived [12]. The supersymmetrization of this construction is not a straightforward task because in many cases adding the interaction with background fields results in the so called "weak supersymmetry" algebra [13].

The program to construct  $N = 4$  supersymmetric extensions of the Karabali and Nair approach was initiated in [14], where a  $N = 4$  supersymmetric mechanics describing the motion of a charged particle over the  $\mathbb{CP}^n$  manifold in the presence of background  $U(n)$  gauge fields was constructed. In contrast with  $N = 4$  supersymmetric mechanics on  $S^4$  [15] which possesses only the  $SO(4)$  invariance, the  $N = 4$  supersymmetric mechanics on  $\mathbb{CP}^n$  is invariant under the full  $SU(n+1)$  group, which of course is realized non-linearly. Thus, the key point is to understand whether the inclusion of non-Abelian  $U(n)$  background gauge fields in  $N = 4$  supersymmetric mechanics on  $\mathbb{CP}^n$  preserves the  $SU(n+1)$  symmetry of the model. In the present paper we explicitly construct the currents, forming the  $su(n+1)$  algebra with respect to a standard Poisson brackets, which commute with the  $N = 4$  supercharges of the model [14]. We also reveal the nice structure of the classical Hamiltonian of  $N = 4$ ,  $\mathbb{CP}^n$  supersymmetric mechanics in the presence of background  $U(n)$  gauge fields to be a direct sum of two Casimir operators: one Casimir operator on the  $SU(n+1)$  group contains our bosonic and fermionic coordinates and momenta, while the second one, on the  $SU(1, n)$  group, is constructed from isospin degrees of freedom only.

The paper is organized as follows. In Section 2 we review the symmetry properties of the bosonic  $\mathbb{CP}^n$  mechanics with and without background Abelian and non-Abelian gauge fields. In Section 3, after a short discussion of the symmetries of  $N = 4$  supersymmetric  $\mathbb{CP}^n$  model in the absence of background fields, we present the explicit form of the  $su(n+1)$  currents commuting with the supercharges. The summary of our results and perspectives for future studying is collected in the Conclusion.

## 2 Preliminaries: Bosonic $\mathbb{CP}^n$ mechanics and its symmetries

### 2.1 Bosonic $\mathbb{CP}^n$ model: Lagrangian approach

The standard bosonic  $\mathbb{CP}^n$  model is defined in terms of  $2n$  bosonic coordinates  $\{z^\alpha, \bar{z}_\alpha, \alpha = 1, \dots, n\}$  depending on time  $t$  by the Lagrangian

$$L = \int dt \mathcal{L} = \int dt g_{\alpha}^{\beta} \dot{z}^{\alpha} \dot{\bar{z}}_{\beta}, \quad (2.1)$$

where the  $\mathbb{CP}^n$  metric  $g_{\alpha}^{\beta}$  has the standard Fubini-Study form

$$g_{\alpha}^{\beta} = \frac{1}{(1 + z \cdot \bar{z})} \left[ \delta_{\alpha}^{\beta} - \frac{\bar{z}_{\alpha} z^{\beta}}{(1 + z \cdot \bar{z})} \right], \quad z \cdot \bar{z} \equiv z^{\alpha} \bar{z}_{\alpha}. \quad (2.2)$$

$\mathbb{CP}^n$  mechanics provides a very simple example of the non-linear realization of  $SU(n+1)$  symmetry in which a  $U(n)$  subgroup is realized linearly [8]. Thus,  $\mathbb{CP}^n$  mechanics could be interpreted as the  $\sigma$ -model on the coset  $SU(n+1)/U(n)$ . The corresponding construction is quite simple.

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<sup>1</sup>Non-compact versions have been considered in [10, 11].

Firstly, let us fix the commutation relations of the  $su(n+1)$  algebra to be

$$\begin{aligned} [\mathbf{R}_\alpha, \bar{\mathbf{R}}^\beta] &= 2 \mathbf{J}_\alpha^\beta, \quad [\mathbf{J}_\alpha^\beta, \mathbf{J}_\gamma^\sigma] = \frac{1}{2} (\delta_\gamma^\beta \mathbf{J}_\alpha^\sigma - \delta_\alpha^\sigma \mathbf{J}_\gamma^\beta), \\ [\mathbf{J}_\alpha^\beta, \mathbf{R}_\gamma] &= \frac{1}{2} (\delta_\gamma^\beta \mathbf{R}_\alpha + \delta_\alpha^\beta \mathbf{R}_\gamma), \quad [\mathbf{J}_\alpha^\beta, \bar{\mathbf{R}}^\gamma] = -\frac{1}{2} (\delta_\alpha^\gamma \bar{\mathbf{R}}^\beta + \delta_\alpha^\beta \bar{\mathbf{R}}^\gamma). \end{aligned} \quad (2.3)$$

Thus, the generators  $\mathbf{R}_\alpha, \bar{\mathbf{R}}^\alpha$  belong to the coset  $SU(n+1)/U(n)$ , while the  $\mathbf{J}_\alpha^\beta$  form  $u(n)$  subalgebra. In addition, these generators are chosen to be hermitian ones

$$(\mathbf{R}_\alpha)^\dagger = \bar{\mathbf{R}}^\alpha, \quad (\mathbf{J}_\alpha^\beta)^\dagger = \mathbf{J}_\beta^\alpha. \quad (2.4)$$

Now, we can realize the action of  $SU(n+1)$  group on the  $SU(n+1)/U(n)$  coset element  $g$

$$g = e^{i(x^\alpha \mathbf{R}_\alpha + \bar{x}_\alpha \bar{\mathbf{R}}^\alpha)} \quad (2.5)$$

by the left multiplications as

$$g_0 g = e^{i(a^\alpha \mathbf{R}_\alpha + \bar{a}_\alpha \bar{\mathbf{R}}^\alpha + b_\alpha^\beta \mathbf{J}_\beta^\alpha)} e^{i(x^\alpha \mathbf{R}_\alpha + \bar{x}_\alpha \bar{\mathbf{R}}^\alpha)} = e^{i(x'^\alpha \mathbf{R}_\alpha + \bar{x}'_\alpha \bar{\mathbf{R}}^\alpha)} h, \quad (2.6)$$

where  $h \in U(n)$ . The explicit form of  $\{\mathbf{R}_\alpha, \bar{\mathbf{R}}^\alpha\}$  transformations reads

$$z'^\alpha = z^\alpha + a^\alpha + (z \cdot \bar{a}) z^\alpha, \quad \bar{z}'_\alpha = \bar{z}_\alpha + \bar{a}_\alpha + (a \cdot \bar{z}) \bar{z}_\alpha, \quad (2.7)$$

where  $\{z^\alpha, \bar{z}_\alpha\}$  are defined as

$$z^\alpha \equiv \frac{\tan \sqrt{x \cdot \bar{x}}}{\sqrt{x \cdot \bar{x}}} x^\alpha, \quad \bar{z}_\alpha \equiv \frac{\tan \sqrt{x \cdot \bar{x}}}{\sqrt{x \cdot \bar{x}}} \bar{x}_\alpha. \quad (2.8)$$

One may easily check that the Lagrangian (2.1) is indeed invariant under transformations (2.7).

The basic covariant objects in the coset approach are the Cartan forms

$$g^{-1} dg = i dz^\alpha e_\alpha^\beta \mathbf{R}_\beta + i \bar{\mathbf{R}}^\alpha e_\alpha^\beta d\bar{z}_\beta + 2(z^\alpha \omega_\beta^\gamma d\bar{z}_\gamma - dz^\gamma \omega_\gamma^\alpha \bar{z}_\beta) \mathbf{J}_\alpha^\beta. \quad (2.9)$$

With our definitions the explicit expressions for the vielbeins  $e_\alpha^\beta$  and  $U(n)$ -connections  $\omega_\alpha^\beta$  on the  $\mathbb{CP}^n$  manifold entering (2.9) read [16]

$$e_\alpha^\beta = \frac{1}{\sqrt{1+z \cdot \bar{z}}} \left[ \delta_\alpha^\beta - \frac{\bar{z}_\alpha z^\beta}{\sqrt{1+z \cdot \bar{z}} (1 + \sqrt{1+z \cdot \bar{z}})} \right], \quad (2.10)$$

$$\omega_\alpha^\beta = \frac{1}{\sqrt{1+z \cdot \bar{z}} (1 + \sqrt{1+z \cdot \bar{z}})} \left[ \delta_\alpha^\beta - \frac{\bar{z}_\alpha z^\beta}{2 \sqrt{1+z \cdot \bar{z}} (1 + \sqrt{1+z \cdot \bar{z}})} \right]. \quad (2.11)$$

The vielbeins  $e_\alpha^\beta$  define the  $SU(n+1)$  covariant derivatives of our "fields"  $\{z^\alpha, \bar{z}_\alpha\}$

$$\nabla_t z^\alpha = \dot{z}^\beta e_\beta^\alpha, \quad \nabla_t \bar{z}_\alpha = e_\alpha^\gamma \dot{\bar{z}}_\gamma \quad (2.12)$$

and the  $SU(n+1)$  invariant Lagrangian density unambiguously restored to be

$$\mathcal{L} = \nabla_t z^\alpha \nabla_t \bar{z}_\alpha = g_\alpha^\beta \dot{z}^\alpha \dot{\bar{z}}_\beta, \quad (2.13)$$

which coincides with (2.1).

## 2.2 Bosonic $\mathbb{CP}^n$ model: Hamiltonian approach

The Lagrangian approach we considered in the previous Subsection has a nice geometric interpretation within the coset construction. In contrast, the Hamiltonian approach is more flexible and provides a simple possibility to introduce the interactions with the background gauge fields in the pure  $\mathbb{CP}^n$  mechanics.

### 2.2.1 $\mathbb{CP}^n$ model

The Hamiltonian of the  $\mathbb{CP}^n$  models (2.1) reads

$$H = \bar{p}^\alpha (g^{-1})_\alpha{}^\beta p_\beta, \quad (2.14)$$

with

$$(g^{-1})_\alpha{}^\beta = (1 + z \cdot \bar{z}) [\delta_\alpha^\beta + \bar{z}_\alpha z^\beta]. \quad (2.15)$$

The  $SU(n+1)$  invariance of the  $\mathbb{CP}^n$  model at the Hamiltonian level means that the Hamiltonian (2.14) has vanishing Poisson brackets with the currents

$$R_\alpha = p_\alpha + \bar{z}_\alpha \bar{z}_\beta \bar{p}^\beta, \quad \bar{R}^\alpha = \bar{p}^\alpha + z^\alpha z^\beta p_\beta, \quad J_\alpha{}^\beta = \frac{i}{2} (z^\beta p_\alpha - \bar{z}_\alpha \bar{p}^\beta) + \frac{i}{2} \delta_\alpha^\beta (z^\gamma p_\gamma - \bar{z}_\gamma \bar{p}^\gamma). \quad (2.16)$$

These currents form the  $su(n+1)$  algebra

$$\begin{aligned} \{R_\alpha, \bar{R}^\beta\} &= 2i J_\alpha{}^\beta, \quad \{J_\alpha{}^\beta, J_\gamma{}^\delta\} = \frac{i}{2} (\delta_\gamma^\beta J_\alpha{}^\delta - \delta_\alpha^\delta J_\gamma{}^\beta), \\ \{J_\alpha{}^\beta, R_\gamma\} &= \frac{i}{2} (\delta_\gamma^\beta R_\alpha + \delta_\alpha^\beta R_\gamma), \quad \{J_\alpha{}^\beta, \bar{R}^\gamma\} = -\frac{i}{2} (\delta_\alpha^\gamma \bar{R}^\beta + \delta_\beta^\gamma \bar{R}^\alpha) \end{aligned} \quad (2.17)$$

with respect to the standard Poisson brackets

$$\{z^\alpha, p_\beta\} = \delta_\beta^\alpha, \quad \{\bar{z}_\alpha, \bar{p}^\beta\} = \delta_\alpha^\beta. \quad (2.18)$$

Moreover, one may check that the Hamiltonian (2.14) coincides with the quadratic  $su(n+1)$  Casimir operator

$$\mathcal{C} = R_\alpha \bar{R}^\alpha + 2J_\alpha{}^\beta J_\beta{}^\alpha - \frac{2}{n+1} J_\alpha{}^\alpha J_\beta{}^\beta, \quad (2.19)$$

with the realization (2.16) taken into account. This is just another visualization of the  $SU(n+1)$  invariance of  $\mathbb{CP}^n$  model.

### 2.2.2 Constant magnetic field

If we are interested in the introduction of some interaction in our system which preserves the  $SU(n+1)$  symmetry, then we should think about the interaction with background magnetic fields only, because it is impossible to construct any  $SU(n+1)$  invariant potential term from our coordinates  $\{z^\alpha, \bar{z}_\alpha\}$  which transform as in (2.7). Moreover, in accordance with the  $SU(n+1)/U(n)$  interpretation of  $\mathbb{CP}^n$  mechanics, in order to preserve all symmetries, the background fields could be either Abelian (which corresponds to the  $U(1)$  group in the stability subgroup  $U(n)$ ) or non-Abelian, living on the whole  $U(n)$  subgroup.

The simplest way to introduce the interaction with the Abelian  $U(1)$  background magnetic field  $B$  is to modify the  $U(n)$  currents  $J_\alpha{}^\beta$  (2.16) as

$$\tilde{J}_\alpha{}^\beta = J_\alpha{}^\beta + B \delta_\alpha^\beta. \quad (2.20)$$

It is a matter of straightforward calculations to find new coset  $SU(n+1)/U(n)$  currents  $\{\tilde{R}_\alpha, \tilde{\bar{R}}^\alpha\}$

$$\tilde{R}_\alpha = R_\alpha + iB \bar{z}_\alpha, \quad \tilde{\bar{R}}^\alpha = \bar{R}^\alpha - iB z^\alpha, \quad (2.21)$$

which form, together with  $\tilde{J}_\alpha{}^\beta$ , the same  $su(n+1)$  algebra (2.17).

If we now define the Hamiltonian as the Casimir operator (2.19) constructed from the generators (2.21), then we will get

$$H = \tilde{\bar{p}}^\alpha (g^{-1})_\alpha{}^\beta \tilde{p}_\beta + \frac{2n}{n+1} B^2, \quad (2.22)$$

where

$$\begin{cases} \tilde{p}_\alpha = p_\alpha - iB \frac{\bar{z}_\alpha}{1+z \cdot \bar{z}} \\ \tilde{\bar{p}}^\alpha = \bar{p}^\alpha + iB \frac{z^\alpha}{1+z \cdot \bar{z}} \end{cases} \Rightarrow \{\tilde{p}_\alpha, \tilde{\bar{p}}^\beta\} = -2iB g_\alpha{}^\beta. \quad (2.23)$$

Clearly, the Hamiltonian (2.22) describes  $\mathbb{CP}^n$  mechanics in the uniform  $U(1)$  background.

One can include also some analog of the oscillator potential fields, which naturally breaks the full  $SU(n+1)$  symmetry down to  $U(n)$ , but nonetheless preserves the exact solvability and all  $\mathbb{CP}^n$  oscillator symmetries, including hidden ones, even in the presence of a constant magnetic field [17].

### 2.2.3 Non-Abelian magnetic field

The same strategy we used in the previous Subsection can be applied for the more interesting case of non-Abelian  $U(n)$  background.

Similarly to the case of  $U(1)$  background, we will start with the modification of  $U(n)$  currents  $J_\alpha^\beta$  (2.16) now as

$$\tilde{J}_\alpha^\beta = J_\alpha^\beta + \hat{J}_\alpha^\beta. \quad (2.24)$$

Here, the currents  $\hat{J}_\alpha^\beta$  have vanishing brackets with the coordinates and momenta  $\{z^\alpha, \bar{z}_\alpha, p_\alpha, \bar{p}^\alpha\}$  and form the same  $U(n)$  algebra (2.17)

$$\{\hat{J}_\alpha^\beta, \hat{J}_\gamma^\delta\} = \frac{i}{2} \left( \delta_\gamma^\beta \hat{J}_\alpha^\delta - \delta_\alpha^\delta \hat{J}_\gamma^\beta \right). \quad (2.25)$$

Now, one has to construct the new  $SU(n+1)/U(n)$  currents which will span, together with  $\tilde{J}_\alpha^\beta$  (2.24), the  $su(n+1)$  algebra. After some calculations one may find these modified  $su(n+1)$  generators

$$\begin{aligned} \tilde{R}_\alpha &= R_\alpha + \frac{2i}{(1 + \sqrt{1 + z \cdot \bar{z}})} \hat{J}_\alpha^\beta \bar{z}_\beta + \frac{i}{(1 + \sqrt{1 + z \cdot \bar{z}})^2} \bar{z}_\alpha z^\beta \hat{J}_\beta^\gamma \bar{z}_\gamma, \\ \bar{\tilde{R}}^\alpha &= \bar{R}^\alpha - \frac{2i}{(1 + \sqrt{1 + z \cdot \bar{z}})} z^\beta \hat{J}_\beta^\alpha - \frac{i}{(1 + \sqrt{1 + z \cdot \bar{z}})^2} z^\alpha z^\beta \hat{J}_\beta^\gamma \bar{z}_\gamma, \\ \tilde{J}_\alpha^\beta &= -\frac{i}{2} \left\{ \tilde{R}_\alpha, \bar{\tilde{R}}^\beta \right\} = J_\alpha^\beta + \hat{J}_\alpha^\beta. \end{aligned} \quad (2.26)$$

The corresponding Hamiltonian can be again defined as the Casimir operator (2.19) constructed now from the currents (2.26). Explicitly it reads

$$H = \bar{\tilde{p}}^\alpha (g^{-1})_\alpha^\beta \tilde{p}_\beta + 2 \hat{J}_\alpha^\beta \hat{J}_\beta^\alpha - \frac{2}{n+1} \hat{J}_\alpha^\alpha \hat{J}_\beta^\beta, \quad (2.27)$$

where now

$$\begin{cases} \tilde{p}_\alpha = p_\alpha - 2i \omega_\alpha^\beta \hat{J}_\beta^\gamma \bar{z}_\gamma \\ \bar{\tilde{p}}^\alpha = \bar{p}^\alpha + 2i z^\gamma \hat{J}_\gamma^\beta \omega_\beta^\alpha \end{cases} \Rightarrow \{ \tilde{p}_\alpha, \bar{\tilde{p}}^\beta \} = -2i e_\alpha^\mu e_\nu^\beta \hat{J}_\mu^\nu, \quad (2.28)$$

with vielbeins  $e_\alpha^\beta$  and  $U(n)$  connections  $\omega_\alpha^\beta$  on  $SU(n+1)/U(n)$  defined in (2.10), (2.11), respectively.

Let us note that the  $\hat{J}_\alpha^\beta \rightarrow B\delta_\alpha^\beta$  reduction brings us from the non-Abelian to the  $U(1)$  Abelian case. Another comment concerns the structure of the Hamiltonian (2.27) which can be represented as

$$H = \bar{\tilde{p}}^\alpha (g^{-1})_\alpha^\beta \tilde{p}_\beta + 2 \mathcal{C}_{U(n)} \quad (2.29)$$

where

$$\mathcal{C}_{U(n)} = \hat{J}_\alpha^\beta \hat{J}_\beta^\alpha - \frac{1}{n+1} \hat{J}_\alpha^\alpha \hat{J}_\beta^\beta \quad (2.30)$$

is just the  $U(n)$  Casimir operator constructed from  $U(n)$  currents  $\hat{J}_\beta^\alpha$ . It is evident that  $\mathcal{C}_{U(n)}$  commutes with all  $SU(n+1)$  currents (2.24), (2.26). So, from the symmetry point of view one may consider on the same footing the following Hamiltonian

$$\tilde{H} = \bar{\tilde{p}}^\alpha (g^{-1})_\alpha^\beta \tilde{p}_\beta + \gamma \mathcal{C}_{U(n)}, \quad (2.31)$$

where  $\gamma$  is an arbitrary constant. This means that one should use another argument to select the Hamiltonian with some fixed value of  $\gamma$ . In principle, one may even consider the reduced system with the fixed value of the Casimir  $\mathcal{C}_{U(n)}$ .

Finally, let us stress that the structure of the currents  $\hat{J}_\beta^\alpha$  is completely irrelevant for the present construction: all that we need is to be sure that these currents span the  $U(n)$  algebra (2.25). So, one may choose these currents to be constructed from some additional isospin degrees of freedom, or one may introduce new physical bosonic coordinates and their momenta to realize these currents. In this case we will have the system extending standard  $\mathbb{CP}^n$  mechanics.

### 3 N=4 Supersymmetry

In this Section we will extend the consideration from the previous Section to the case of the  $N = 4$  supersymmetric extension of  $\mathbb{CP}^n$  mechanics.

#### 3.1 N=4 supersymmetric $\mathbb{CP}^n$ model: free case

In order to construct the  $N = 4$  supersymmetric extension of  $\mathbb{CP}^n$  mechanics one should introduce  $4n$  fermionic variables  $\{\psi_i^\alpha, \bar{\psi}_\alpha^i, i = 1, 2\}$  obeying the following Dirac brackets (together with the previously defined brackets (2.18))

$$\begin{aligned} \{\psi_i^\alpha, \bar{\psi}_\beta^j\} &= i\delta_i^j (g^{-1})_\beta^\alpha, & \{p_\alpha, \bar{p}^\beta\} &= -i(g_\alpha^\beta g_\mu^\nu + g_\alpha^\nu g_\mu^\beta) \bar{\psi}_\nu^i \psi_i^\mu, \\ \{p_\alpha, \psi_i^\beta\} &= -\frac{1}{(1+z \cdot \bar{z})} [\bar{z}_\alpha \psi_i^\beta + \delta_\alpha^\beta \psi_i^\gamma \bar{z}_\gamma], & \{\bar{p}^\alpha, \bar{\psi}_\beta^i\} &= -\frac{1}{(1+z \cdot \bar{z})} [z^\alpha \bar{\psi}_\beta^i + \delta_\beta^\alpha z^\gamma \bar{\psi}_\gamma^i]. \end{aligned} \quad (3.1)$$

Now, it is not too hard to check that the supercharges  $Q^i, \bar{Q}_i$  have the extremely simple form [5, 6, 7]

$$Q^i = \bar{p}^\alpha \bar{\psi}_\alpha^i, \quad \bar{Q}_i = \psi_i^\alpha p_\alpha. \quad (3.2)$$

They are perfectly anticommute (in virtue of (3.1), (2.18)) as

$$\{Q^i, \bar{Q}_j\} = i\delta_j^i H, \quad \{Q^i, Q^j\} = \{\bar{Q}_i, \bar{Q}_j\} = 0, \quad (3.3)$$

where the Hamiltonian  $H$  reads <sup>2</sup>

$$H = \bar{p}^\alpha (g^{-1})_\alpha^\beta p_\beta + \frac{1}{4} (g_\mu^\alpha g_\rho^\sigma + g_\mu^\sigma g_\rho^\alpha) \bar{\psi}_{\alpha i} \bar{\psi}_\sigma^i \psi^{\rho j} \psi_j^\mu. \quad (3.4)$$

In the supersymmetric case one may again construct the currents spanning the  $su(n+1)$  algebra (2.17) as<sup>3</sup>

$$\begin{aligned} R_\alpha &= p_\alpha + \bar{z}_\alpha \bar{z}_\beta \bar{p}^\beta - \frac{i}{(1+z \cdot \bar{z})^2} \left( \bar{z}_\alpha \psi_i^\beta \bar{\psi}_\beta^i + \bar{z}_\beta \psi_i^\beta \bar{\psi}_\alpha^i - \frac{2}{1+z \cdot \bar{z}} \bar{z}_\alpha \bar{z}_\beta z^\gamma \psi_i^\beta \bar{\psi}_\gamma^i \right), \\ \bar{R}^\alpha &= \bar{p}^\alpha + z^\alpha z^\beta p_\beta + \frac{i}{(1+z \cdot \bar{z})^2} \left( z^\alpha \psi_i^\beta \bar{\psi}_\beta^i + z^\beta \psi_i^\alpha \bar{\psi}_\beta^i - \frac{2}{1+z \cdot \bar{z}} z^\alpha \bar{z}_\beta z^\gamma \psi_i^\beta \bar{\psi}_\gamma^i \right), \\ J_\alpha^\beta &= -\frac{i}{2} \{R_\alpha, \bar{R}^\beta\}. \end{aligned} \quad (3.5)$$

One may check that the Hamiltonian (3.4) and the supercharges (3.2) have vanishing brackets with the generators of the  $su(n+1)$  algebra (3.5):

$$\{R_\alpha, H\} = \{\bar{R}^\alpha, H\} = 0, \quad \{R_\alpha, Q^i\} = \{\bar{R}^\alpha, Q^i\} = 0, \quad \{R_\alpha, \bar{Q}_i\} = \{\bar{R}^\alpha, \bar{Q}_i\} = 0. \quad (3.6)$$

This result is expected. A less expected statement is that the Hamiltonian (3.4) coincides with the Casimir operator of the  $su(n+1)$  algebra

$$H = R_\alpha \bar{R}^\alpha + 2J_\alpha^\beta J_\beta^\alpha - \frac{2}{n+1} J_\alpha^\alpha J_\beta^\beta, \quad (3.7)$$

with the currents  $\{R_\alpha, \bar{R}^\alpha, J_\alpha^\beta\}$  defined in (3.5). Thus, the  $N = 4$  supersymmetric  $\mathbb{CP}^n$  mechanics has the same symmetry properties as its bosonic core. The only (but crucial) difference is another realization of the  $su(n+1)$  currents (3.5) which includes now additional fermionic degrees of freedom.

<sup>2</sup>The  $su(2)$  indices are raised and lowered as  $A_i = \varepsilon_{ij} A^j, A^i = \varepsilon^{ij} A_j$  with  $\varepsilon_{12} = \varepsilon^{21} = 1$ .

<sup>3</sup>The explicit expression for generators  $J_\alpha^\beta$  is not so illuminating and it can be easily obtained from the definition in (3.5), if needed.

### 3.2 N=4 supersymmetric $\mathbb{CP}^n$ model: interaction

In this Section, based on the approach presented in the previous Section, we will analyze the symmetry of the  $N = 4$  supersymmetric  $\mathbb{CP}^n$  model in the background  $U(n)$  fields.

In order to introduce the interaction in  $N = 4$  supersymmetric  $\mathbb{CP}^n$  mechanics, following [14], we will couple our model with additional currents  $\{\mathcal{R}_\alpha, \overline{\mathcal{R}}^\alpha, \mathcal{J}_\alpha^\beta\}$  spanning  $SU(1, n)$  groups:

$$\begin{aligned}\{\mathcal{R}_\alpha, \overline{\mathcal{R}}^\beta\} &= -2i\mathcal{J}_\alpha^\beta, \\ \{\mathcal{J}_\alpha^\beta, \mathcal{R}_\gamma\} &= \frac{i}{2}(\delta_\gamma^\beta \mathcal{R}_\alpha + \delta_\alpha^\beta \mathcal{R}_\gamma), \quad \{\mathcal{J}_\alpha^\beta, \overline{\mathcal{R}}^\gamma\} = -\frac{i}{2}(\delta_\alpha^\gamma \overline{\mathcal{R}}^\beta + \delta_\alpha^\beta \overline{\mathcal{R}}^\gamma), \\ \{\mathcal{J}_\alpha^\beta, \mathcal{J}_\gamma^\delta\} &= \frac{i}{2}(\delta_\gamma^\beta \mathcal{J}_\alpha^\delta - \delta_\alpha^\delta \mathcal{J}_\gamma^\beta).\end{aligned}\tag{3.8}$$

The model is completely defined by supercharges forming the  $N = 4$  super Poincare algebra (3.3). Such supercharges have been constructed in [14] as

$$Q^i = \bar{p}^\alpha \bar{\psi}_\alpha^i + 2i z^\gamma \mathcal{J}_\gamma^\beta \omega_\beta^\alpha \bar{\psi}_\alpha^i + i \psi^{i\alpha} e_\alpha^\beta \mathcal{R}_\beta, \quad \overline{Q}_i = \psi_i^\alpha p_\alpha - 2i \psi_i^\alpha \omega_\alpha^\beta \mathcal{J}_\beta^\gamma \bar{z}_\gamma + i \overline{\mathcal{R}}^\beta e_\beta^\alpha \bar{\psi}_{i\alpha}, \tag{3.9}$$

where  $e_\alpha^\beta$  and  $\omega_\alpha^\beta$  are the vielbeins and  $U(n)$ -connections on the  $\mathbb{CP}^n \sim SU(n+1)/U(n)$  manifold defined in (2.10) and (2.11), correspondingly.

These supercharges are perfectly anticommuting to span the  $N = 4$  super Poincare algebra (3.3) where the Hamiltonian  $H$  now reads

$$\begin{aligned}H &= (\bar{p} g^{-1} p) - 2i [(\bar{p} g^{-1} \omega \mathcal{J} \bar{z}) - (z \mathcal{J} \omega g^{-1} p)] + (\overline{\mathcal{R}} \mathcal{R}) + 4(z \mathcal{J} \omega g^{-1} \omega \mathcal{J} \bar{z}) \\ &- 2(\psi_i e \mathcal{J} e \bar{\psi}^i) + \frac{1}{4}(g_\mu^\alpha g_\rho^\sigma + g_\mu^\sigma g_\rho^\alpha) \bar{\psi}_\alpha^i \bar{\psi}_\sigma^j \psi_j^\mu.\end{aligned}\tag{3.10}$$

One of the most interesting features of the supercharges (3.9) and Hamiltonian (3.10) is the presence of the full set of  $su(1, n)$  currents (3.8). If we believe that this Hamiltonian is  $su(n+1)$  invariant, than we have to find the corresponding currents spanning  $su(n+1)$  algebra and having the vanishing brackets with the Hamiltonian and supercharges. These new extended  $su(n+1)$  currents can contain, besides our bosonic and fermionic coordinates and momenta, only currents  $\mathcal{J}_\gamma^\delta$  spanning the  $u(n)$  algebra. It is clear that the modified  $su(n+1)$  generators cannot contain the currents  $\{\mathcal{R}_\alpha, \overline{\mathcal{R}}^\beta\}$  - it is just impossible to construct new  $su(n+1)$  generators from our coordinates, momenta and  $su(1, n)$  currents. This means that the Hamiltonian (3.10) cannot be just a Casimir operator of some  $SU(n+1)$  group. It should have a more interesting structure. The better understand the structure of the Hamiltonian (3.10) one may rewrite the term  $\overline{\mathcal{R}} \mathcal{R}$  in (3.10) as

$$\overline{\mathcal{R}}^\alpha \mathcal{R}_\alpha = \mathcal{C}_{su(1, n)} + \left(2\mathcal{J}_\alpha^\beta \mathcal{J}_\beta^\alpha - \frac{2}{n+1} \mathcal{J}_\alpha^\alpha \mathcal{J}_\beta^\beta\right), \tag{3.11}$$

where the  $su(1, n)$  Casimir operator  $\mathcal{C}_{su(1, n)}$  reads

$$\mathcal{C}_{su(1, n)} = \overline{\mathcal{R}}^\alpha \mathcal{R}_\alpha - 2\mathcal{J}_\alpha^\beta \mathcal{J}_\beta^\alpha + \frac{2}{n+1} \mathcal{J}_\alpha^\alpha \mathcal{J}_\beta^\beta. \tag{3.12}$$

Thus we see that the unexpected generators  $\{\mathcal{R}_\alpha, \overline{\mathcal{R}}^\beta\}$  belonging to the coset  $SU(1, n)/U(n)$  enter the Hamiltonian only through the  $su(1, n)$  Casimir operator  $\mathcal{C}_{su(1, n)}$ . The remaining terms depend only on the  $u(n)$  currents  $\mathcal{J}_\alpha^\beta$ . Therefore, we expect that the Hamiltonian (3.10) can be represented as a sum of two Casimir operators

$$H = \tilde{\mathcal{C}}_{su(n+1)} + \mathcal{C}_{su(1, n)}, \tag{3.13}$$

where the Casimir operator  $\tilde{\mathcal{C}}_{su(n+1)}$  has to be constructed from our coordinates, momenta and additional  $u(n)$  currents  $\mathcal{J}_\gamma^\delta$  only. Now we are going to prove this statement.

In order to find these new  $su(n+1)$  currents, which commute with the supercharges (3.9), we note that the structure of new  $u(n)$  generators is obvious: there is no other choice besides the direct sum of generators

$$\tilde{\mathcal{J}}_\alpha^\beta = J_\alpha^\beta + \mathcal{J}_\alpha^\beta. \tag{3.14}$$

Here, the current  $J_\alpha^\beta$  is the same as in the non-interacting  $N = 4$  supersymmetric case (3.5). The structure of the coset  $SU(n+1)/U(n)$  generators is more involved. In order to find them, one has to consider the most general Ansatz for these generators compatible with the explicit  $U(n)$  symmetry

$$\begin{aligned}\tilde{R}_\alpha &= R_\alpha + i f_1 \mathcal{J}_\alpha^\beta \bar{z}_\beta + i f_2 \bar{z}_\alpha z^\beta \mathcal{J}_\beta^\gamma \bar{z}_\gamma + i f_3 \bar{z}_\alpha \mathcal{J}_\beta^\beta, \\ \bar{\tilde{R}}^\alpha &= \bar{R}^\alpha - i f_1 z^\beta \mathcal{J}_\beta^\alpha - i f_2 z^\alpha z^\beta \mathcal{J}_\beta^\gamma \bar{z}_\gamma - i f_3 z^\alpha \mathcal{J}_\beta^\beta,\end{aligned}\tag{3.15}$$

where the currents  $\{R_\alpha, \bar{R}^\alpha\}$  were defined in (3.5) and the unknown functions  $f_1, f_2, f_3$  depend only on  $(z \cdot \bar{z})$ . Then one has to check that: a) these currents span the  $su(n+1)$  algebra, b) these currents commute with the supercharges (3.9). After a quite lengthy calculation, one may find these function to be

$$f_1 = \frac{2}{(1 + \sqrt{1 + z \cdot \bar{z}})}, \quad f_2 = \frac{1}{(1 + \sqrt{1 + z \cdot \bar{z}})^2}, \quad f_3 = 0,\tag{3.16}$$

and therefore

$$\begin{aligned}\tilde{R}_\alpha &= R_\alpha + \frac{2i}{(1 + \sqrt{1 + z \cdot \bar{z}})} \mathcal{J}_\alpha^\beta \bar{z}_\beta + \frac{i}{(1 + \sqrt{1 + z \cdot \bar{z}})^2} \bar{z}_\alpha z^\beta \mathcal{J}_\beta^\gamma \bar{z}_\gamma, \\ \bar{\tilde{R}}^\alpha &= \bar{R}^\alpha - \frac{2i}{(1 + \sqrt{1 + z \cdot \bar{z}})} z^\beta \mathcal{J}_\beta^\alpha - \frac{i}{(1 + \sqrt{1 + z \cdot \bar{z}})^2} z^\alpha z^\beta \mathcal{J}_\beta^\gamma \bar{z}_\gamma, \\ \tilde{\mathcal{J}}_\alpha^\beta &= -\frac{i}{2} \left\{ \tilde{R}_\alpha, \bar{\tilde{R}}^\beta \right\} = J_\alpha^\beta + \mathcal{J}_\alpha^\beta.\end{aligned}\tag{3.17}$$

All these generators commute with the supercharges  $\{Q^i, \bar{Q}_i\}$  and, therefore, with the Hamiltonian (3.10), while the Hamiltonian itself (3.10) has the suggested structure (3.13). Thus, we conclude that  $N = 4$  supersymmetric  $\mathbb{CP}^n$  mechanics in the background  $U(n)$  fields possesses  $su(n+1)$  symmetry generated by the currents (3.17). Similarly to the bosonic cases, the structure of the  $su(1, n)$  currents  $\{\mathcal{R}_\alpha, \bar{\mathcal{R}}^\alpha, \mathcal{J}_\alpha^\beta\}$  (3.8) is irrelevant for our construction. One may also consider a reduced version of the system by fixing the value of the Casimir operator  $\mathcal{C}_{su(1, n)}$ , which clearly commutes with everything.

## 4 Conclusion

In the present paper we have proved the  $SU(n+1)$  invariance of a  $N = 4$  supersymmetric extension of mechanics describing the motion of a particle over the  $\mathbb{CP}^n$  manifold in the presence of background  $U(n)$  gauge fields. We have explicitly constructed the corresponding  $su(n+1)$  currents which commute with the supercharges and showed that the Hamiltonian of the system can be represented as a direct sum of two Casimir operators. One Casimir operator, on the  $SU(n+1)$  group, contains our bosonic and fermionic coordinates and momenta together with additional  $U(n)$  currents constructed from isospin degrees of freedom, while the second one, on the  $SU(1, n)$  group, contains isospin degrees of freedom only.

Proving the  $SU(n+1)$  invariance of a  $N = 4$  supersymmetric mechanics describing the motion of a charged particle over  $\mathbb{CP}^n$  manifold in the presence of background  $U(n)$  gauge fields [14] is crucial for the possible application of this model to the analysis of the role that the additional fermionic variables play in the quantum Hall effect on  $\mathbb{CP}^n$ . Surely, this is one of the most interesting immediate applications of our results.

The approach we used in this paper, in order to visualize the symmetry of the supersymmetric  $\mathbb{CP}^n$  model, is based mainly on its interpretation as a sigma model on the coset  $SU(n+1)/U(n)$ . In this respect, viewing the  $\mathbb{CP}^n$  manifold as the coset  $Sp(k+1)/U(1) \times Sp(k)$  [18] opens a way not only to turning on the background fields, living on  $U(1) \times Sp(k)$ , but also for the construction of the corresponding supersymmetric extensions. Using  $U(1) \times Sp(k)$  background fields instead of those living on the  $U(n)$  group could bring some new features into QHE on  $\mathbb{CP}^n$  manifold. This will be described in more detail elsewhere.

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